

LOCAL AND GLOBAL EXISTENCE FOR THE LAGRANGIAN AVERAGED NAVIER-STOKES EQUATIONS IN BESOV SPACES

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ABSTRACT. We prove the existence of short time solutions to the incompressible, isotropic Lagrangian Averaged Navier-Stokes equation with low regularity initial data in Besov spaces $B_{p,q}^r(\mathbb{R}^n)$, $r > n/2p$. When $p = 2$ and $n \geq 3$, we obtain global solutions, provided the parameters r, q and n satisfy certain inequalities. This is an improvement over known analogous Sobolev space results, which required $n = 3$.

1. INTRODUCTION

The Lagrangian Averaged Navier-Stokes (LANS) equation is a recently derived approximation to the Navier-Stokes equation. The equation is obtained via an averaging process applied at the Lagrangian level, resulting in a modified energy functional. The geodesics of this energy functional satisfy the Lagrangian Averaged Euler (LAE) equation, and the LANS equation is derived from the LAE equation in an analogous fashion to the derivation of the Navier-Stokes equation from the Euler equation. For an exhaustive treatment of this process, see [8], [9], [3] and [5]. In [6] and [2], the authors discuss the numerical improvements that use of the LANS equation provides over more common approximation techniques of the Navier-Stokes equation.

On a region without boundary, the isotropic, incompressible form of the LANS equation is given by

$$(1.1) \quad \begin{aligned} \partial_t u + (u \cdot \nabla)u + \operatorname{div} \tau^\alpha u &= -(1 - \alpha^2 \Delta)^{-1} \nabla p + \nu \Delta u \\ u &= u(t, x), \operatorname{div} u = 0, u(0, x) = u_0(x), \end{aligned}$$

with the terms defined as follows. First, $u : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ for some time strip $I = [0, T)$ denotes the velocity of the fluid, $\alpha > 0$ is a constant, $p : I \times \mathbb{R}^n \rightarrow \mathbb{R}$ denotes the fluid pressure, $\nu > 0$ is a constant due to the viscosity of the fluid, and $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $\operatorname{div} u_0 = 0$. Next, the differential operators ∇, Δ , and div are spatial differential operators with their standard definitions. The term $(v \cdot \nabla)w$, also denoted $\nabla_v w$, is the vector field with j^{th} component $\sum_{i=1}^n v_i \partial_i w_j$. The Reynolds stress τ^α is given by

$$\tau^\alpha u = \alpha^2 (1 - \alpha^2 \Delta)^{-1} [\operatorname{Def}(u) \cdot \operatorname{Rot}(u)]$$

where $\operatorname{Rot}(u) = (\nabla u - \nabla u^T)/2$ and $\operatorname{Def}(u) = (\nabla u + \nabla u^T)/2$. We remark that setting $\alpha = 0$ in equation (1.1) recovers the Navier-Stokes equation.

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There is a wide variety of local existence results for the LANS equation in various settings, including [8], [3], [4] and [7]. In [4], Marsden and Shkoller proved the existence of global solutions to the LANS equation with initial data in the Sobolev space $H^{3,2}(\mathbb{R}^3)$. In [7], this result was improved, achieving global existence for data in the space $H^{3/4,2}(\mathbb{R}^3)$. In this article we seek local and global solutions to the LANS equation with initial data in the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$, with an emphasis placed on minimizing s . These results are comparable to those in [7] for Sobolev space initial data.

The main distinction between the results in this paper and those in [7] is that the extension of local results to global results in the Besov space setting does not follow from standard energy methods, requiring significantly more delicate Fourier analysis. As a result of this argument, the global existence result can be applied in dimension $n \geq 3$, which improves the requirement $n = 3$ in [7].

The paper is organized as follows. We devote the rest of this section to defining solution spaces and stating our main theorems. Theorem 1 yields an *a priori* bound for solutions to the LANS equation, and, as mentioned in the preceding paragraph, is the main result of the paper. Its proof can be found in Section 3. Theorems 2 through 5 are local existence theorems, and Corollary 1 is the global existence result. In Section 2, we recall some known Besov space operator estimates which will be useful in the proofs of our main results. In Section 4 we prove Theorems 2 and 4 and in Section 5 we prove Theorems 3 and 5.

As mentioned above, we denote Besov spaces by $B_{p,q}^s(\mathbb{R}^n)$, with norm denoted by $\|\cdot\|_{B_{p,q}^s} = \|\cdot\|_{s,p,q}$ (a complete definition of these spaces can be found in Section 2). We define the space

$$C_{a;s,p,q}^T = \{f \in C((0, T) : B_{p,q}^s(\mathbb{R}^n)) : \|f\|_{a;s,p,q} < \infty\},$$

where

$$\|f\|_{a;s,p,q} = \sup\{t^a \|f(t)\|_{s,p,q} : t \in (0, T)\},$$

$T > 0$, $a \geq 0$, and $C(A : B)$ is the space of continuous functions from A to B . We let $\dot{C}_{a;s,p,q}^T$ denote the subspace of $C_{a;s,p,q}^T$ consisting of f such that

$$\lim_{t \rightarrow 0^+} t^a f(t) = 0 \text{ (in } B_{p,q}^s(\mathbb{R}^n)\text{)}.$$

Note that while the norm $\|\cdot\|_{a;s,p,q}$ lacks an explicit reference to T , there is an implicit T dependence. We also say $u \in BC(A : B)$ if $u \in C(A : B)$ and $\sup_{a \in A} \|u(a)\|_B < \infty$. Lastly, setting $\mathbb{M}((0, T) : \mathbb{E})$ to be the set of measurable functions defined on $(0, T)$ with values in the space \mathbb{E} , we define

$$L^a((0, T) : B_{p,q}^s(\mathbb{R}^n)) = \{f \in \mathbb{M}((0, T) : B_{p,q}^s(\mathbb{R}^n)) : (\int_0^T \|f(t)\|_{s,p,q}^a dt)^{1/a} < \infty\}.$$

As a last bit of notation, we set r^+ to be a real number arbitrarily close to, but strictly greater than, r . Similarly, r^- is arbitrarily close to, but strictly less than, r .

Now we state our main theorems. Our first result provides a uniform-in-time bound on the Besov norm of solutions to the LANS equation.

Theorem 1. *Let $u : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a solution to (1.1) on a time strip $I = [0, T)$. Then*

$$\|u(t)\|_{B_{2,q}^r} \leq \|u_0\|_{B_{2,q}^r} \exp \left(C \int_I \|u(t)\|_{B_{2,q}^{1+n/2}} dt \right),$$

where $r > 2$.

We now state our four local existence theorems. As we are working in low regularity spaces, these local solutions are not classical solutions. Instead, they are solutions to an integral equation derived from the classical formulation of the PDE through Duhamel's principle. A more explicit accounting of this can be found in Section 4. Finally, all of these local solutions depend uniquely and continuously on the initial data.

Theorem 2. *Let $s = 3/4$ and let $u_0 \in B_{2,q}^{s+}(\mathbb{R}^3)$ be divergence-free. Then there exists a local solution u to the LANS equation (1.1), where*

$$(1.2) \quad u \in BC([0, T) : B_{2,q}^{s+}(\mathbb{R}^3)) \cap \dot{C}_{(1+s^+)/2; 1^+, 2, q}^T,$$

and T is a non-increasing function of $\|u_0\|_{s^+, 2, q}$, with $T = \infty$ if $\|u_0\|_{s^+, 2, q}$ is sufficiently small.

Next, let $p > n \geq 3$ and let $s = n/2p$. Then for any divergence-free $u_0 \in B_{p,q}^{s+}(\mathbb{R}^n)$, there exists a locally well-posed solution u to the LANS equation (1.1), where

$$u \in BC([0, T) : B_{p,q}^{s+}(\mathbb{R}^n)) \cap \dot{C}_{(1+s^+)/2; 1^+, p, q}^T,$$

and T is a non-increasing function of $\|u_0\|_{s^+, 2, q}$, with $T = \infty$ if $\|u_0\|_{s^+, 2, q}$ is sufficiently small.

Our third theorem is analogous to the second, with the space $L^a((0, T) : B_{p,q}^s(\mathbb{R}^n))$ replacing $\dot{C}_{a; s, p, q}^T$.

Theorem 3. *Let $s = 3/4$ and let $u_0 \in B_{2,q}^{s_1}(\mathbb{R}^3)$ be divergence-free. Then there exists a local solution u to the LANS equation (1.1), where*

$$(1.3) \quad u \in BC([0, T) : B_{2,q}^{s_1}(\mathbb{R}^n)) \cap L^{2/(1+s^+)}((0, T) : B_{2,q}^{1^+}(\mathbb{R}^n)),$$

and T is a non-increasing function of $\|u_0\|_{s^+, 2, q}$, with $T = \infty$ if $\|u_0\|_{s^+, 2, q}$ is sufficiently small.

Next, let $p > n \geq 3$ and $s = n/2p$. Then for any divergence-free $u_0 \in B_{p,q}^{s+}(\mathbb{R}^n)$ there exists a local solution u to the LANS equation (1.1) where

$$u \in BC([0, T) : B_{p,q}^{s+}(\mathbb{R}^n)) \cap L^{2/(1+s^+)}((0, T) : B_{p,q}^{1^+}(\mathbb{R}^n)),$$

and T is a non-increasing function of $\|u_0\|_{s^+, 2, q}$, with $T = \infty$ if $\|u_0\|_{s^+, 2, q}$ is sufficiently small.

We note that the use of $(\frac{n}{2p})^+$ in these two theorems is due to the requirement in Proposition 2 that $s > n(2/p_1 - 1/p)$ (with $2p \geq p_1$), which differs from the analogous Sobolev space requirement that $s \geq n(2/p_1 - 1/p)$.

In the next two theorems, the minimal regularity achieved is $(n/p)^+$. However, these two theorems are significantly more general, allowing for a much wider range of parameter values, specifically with regard to the auxiliary space.

Theorem 4. *For any divergence-free $u_0 \in B_{p,q}^r(\mathbb{R}^n)$ there exists a local solution to (1.1) such that*

$$(1.4) \quad u \in BC([0, T] : B_{p,q}^r) \cap \dot{C}_{a;s,\tilde{p},q}^T,$$

for some $T > 0$, provided the parameters satisfy the list of conditions (4.6), including that $r < s < r + 1$. The minimal r achieved by these methods is $r = (\frac{n}{p})^+$.

Also, $T = T(\|u_0\|_{r,p,q})$ can be chosen to be a non-increasing function of $\|u_0\|_{r,p,q}$, with $T = \infty$ if $\|u_0\|_{r,p,q}$ is sufficiently small.

Theorem 5. *For any divergence-free $u_0 \in B_{p,q}^r(\mathbb{R}^n)$ there exists a local solution u to (1.1) such that*

$$(1.5) \quad u \in BC([0, T] : B_{p,q}^r) \cap L^\sigma((0, T) : B_{\tilde{p},q}^s),$$

for some $T > 0$, provided the parameters satisfy the list of conditions (5.1), including that $r < s < r + 1$. The minimal r achieved by these methods is $r = (\frac{n}{p})^+$. Also, $T = T(\|u_0\|_{r,p,q})$ can be chosen to be a non-increasing function of $\|u_0\|_{r,p,q}$, with $T = \infty$ if $\|u_0\|_{r,p,q}$ is sufficiently small.

Combining these local existence results with Theorem 1 (which requires Proposition 8 or Proposition 13, depending on the context), we get the following corollary.

Corollary 1. *The local solutions from Theorems 2 and 4 given by equations (1.2) and (1.4) (with $p = \tilde{p} = 2$), respectively, can be extended to global solutions. The local solutions from Theorems 3 and 5 given by equations (1.3) and (1.5) (with $p = \tilde{p} = 2$), respectively, can also be extended to global solutions.*

The proof of the corollary follows from standard extension methods.

2. BESOV SPACES

We begin by defining the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$. Let $\psi_0 \in \mathcal{S}$ be an even, radial function such that $\hat{\psi}_0 \geq 0$, the support of $\hat{\psi}_0$ lies in the annulus $A_0 := \{\xi \in \mathbb{R}^n : 2^{-1} < |\xi| < 2\}$, and $\sum_{j \in \mathbb{Z}} \hat{\psi}_0(2^{-j}\xi) = 1$ for all $\xi \neq 0$. We then define $\hat{\psi}_j(\xi) = 2^{jn}\hat{\psi}_0(2^{-j}\xi)$, and remark that $\hat{\psi}_j$ is supported in $A_j := \{\xi \in \mathbb{R}^n : 2^{j-1} < |\xi| < 2^{j+1}\}$. We also define Ψ by

$$\hat{\Psi}(\xi) = 1 - \sum_{k=0}^{\infty} \hat{\psi}_k(\xi).$$

We define the operators Δ_j and S_j by

$$\Delta_j f = \psi_j * f, \quad S_j f = \sum_{k=-\infty}^j \Delta_k f,$$

and record some properties of these operators. Applying the Fourier Transform and recalling that $\hat{\psi}_j$ is supported on $2^{j-1} \leq |\xi| \leq 2^{j+1}$, it follows that

$$(2.1) \quad \begin{aligned} \Delta_j \Delta_k f &= 0, & |j - k| \geq 2 \\ \Delta_j (S_{k-3} f \Delta_k g) &= 0 & |j - k| \geq 3, \end{aligned}$$

and, if $|i - k| \leq 1$, then

$$(2.2) \quad \Delta_j(\Delta_k f \Delta_i g) = 0 \quad j > k + 3.$$

For $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$ we define the space $\tilde{B}_{p,q}^s(\mathbb{R}^n)$ to be the set of distributions such that

$$\|u\|_{\tilde{B}_{p,q}^s} = \left(\sum_{j=0}^{\infty} (2^{js} \|\Delta_j u\|_{L^p})^q \right)^{1/q} < \infty,$$

with the usual modification when $q = \infty$. Finally, we define the Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ by the norm

$$\|f\|_{B_{p,q}^s} = \|\Psi * f\|_p + \|f\|_{\tilde{B}_{p,q}^s},$$

for $s > 0$. For $s > 0$, we define $B_{p',q'}^{-s}$ to be the dual of the space $B_{p,q}^s$, where p', q' are the Holder-conjugates to p, q .

Now we turn our attention to establishing some basic Besov space estimates. First, we let $1 \leq q_1 \leq q_2 \leq \infty$, $\beta_1 \leq \beta_2$, $s > 0$, $1 \leq p_1 \leq p_2 \leq \infty$ and $\gamma_1 = \gamma_2 + n(1/p_1 - 1/p_2)$. Then we have the following:

$$(2.3) \quad \begin{aligned} \|f\|_{B_{p,q_2}^{\beta_1}} &\leq C \|f\|_{B_{p,q_1}^{\beta_2}} \\ \|f\|_{B_{p_2,q}^{\gamma_2}} &\leq C \|f\|_{B_{p_1,q}^{\gamma_1}} \\ \|f\|_{L^p} &\leq \|f\|_{B_{p,q}^s}. \end{aligned}$$

We also have the following fractional Leibniz rule estimate (see, for instance, Lemma 2.2 in [1]).

Proposition 1. *Let $s > 0$ and $q \in [1, \infty]$. Then we have*

$$\|fg\|_{B_{p,q}^s} \leq C(\|f\|_{L^{p_1}} \|g\|_{B_{p_2,q}^s} + \|g\|_{L^{r_1}} \|f\|_{B_{r_2,q}^s})$$

where $p_i, r_i \in [1, \infty]$ and $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$.

Using Proposition 1 and (2.3), we have, for $1/p = 1/p_1 + 1/p_2$ and $p_1 \leq p_2$,

$$\begin{aligned} \|u^2\|_{B_{p,q}^s} &\leq C \|u\|_{L^{p_2}} \|u\|_{B_{p_1,q}^s} \\ &\leq C \|u\|_{B_{p_2,q}^\varepsilon} \|u\|_{B_{p_1,q}^s} \\ &\leq C \|u\|_{B_{p_1,q}^s}^2 \end{aligned}$$

where $\varepsilon > 0$ and $s = \varepsilon + n(2/p_1 - 1/p)$. We record this as a proposition.

Proposition 2. *Let $s > 0$ and $p, p_1, q \in [1, \infty]$. Then*

$$\|u^2\|_{B_{p,q}^s} \leq C \|u\|_{B_{p_1,q}^s}^2$$

where $p < p_1 \leq 2p$ and $s > n(2/p_1 - 1/p)$.

Next we state two Bernstein inequalities (see Appendix A in [10]). We let $\Lambda = \sqrt{-\Delta}$, $\alpha \geq 0$, and $1 \leq p \leq q \leq \infty$. If $\text{supp } \hat{f} \subset \{\xi \in \mathbb{R}^n : |\xi| \leq K2^j\}$ and

$\text{supp } \hat{g} \subset \{\xi \in \mathbb{R}^n : K_1 2^j \leq |\xi| \leq K_2 2^j\}$ for some $K, K_1, K_2 > 0$ and some integer j , then

$$(2.4) \quad \begin{aligned} \|\Lambda^\alpha f\|_q &\leq C 2^{j\alpha+jn(1/p-1/q)} \|f\|_p \\ \tilde{C} 2^{j\alpha+jn(1/p-1/q)} \|f\|_p &\leq \|\Lambda^\alpha f\|_q \leq C 2^{j\alpha+jn(1/p-1/q)} \|f\|_p. \end{aligned}$$

We conclude with some results governing the behavior of the heat kernel on Besov spaces.

Proposition 3. *Let $1 \leq p_1 \leq p_2 < \infty$, $-\infty < s_1 \leq s_2 < \infty$, and let $0 < q < \infty$. Then*

$$\|e^{t\Delta} u\|_{B_{p_2,q}^{s_2}} \leq C t^{-(s_2-s_1+n/p_1-n/p_2)/2} \|u\|_{B_{p_1,q}^{s_1}}.$$

Using the Sobolev space heat kernel estimate, we get

$$\begin{aligned} \|e^{t\Delta} u\|_{B_{p_2,q}^{s_2}} &= \|e^{t\Delta} u\|_{L^{p_2}} + \left(\sum (2^{js_1} \|2^{j(s_2-s_1)} \phi_j * e^{t\Delta} u\|_{L^{p_2}})^q \right)^{1/q} \\ &\leq t^{(n/p_1-n/p_2)/2} \|u\|_{L^{p_1}} + \left(\sum (2^{js_1} \|\phi_j * e^{t\Delta} u\|_{H^{s_2-s_1,p_2}})^q \right)^{1/q} \\ &\leq t^{-(n/p_1-n/p_2)/2} \|\phi * u\|_{L^{p_1}} + t^\sigma \left(\sum (2^{js_1} \|\phi_j * u\|_{L^{p_1}})^q \right)^{1/q} \\ &\leq t^\sigma \|u\|_{B_{p_1,q}^{s_1}}. \end{aligned}$$

where $\sigma = -(s_2 - s_1 + n/p_1 - n/p_2)/2$.

We have an immediate corollary.

Corollary 2. *With the parameters as in Proposition 3, we have*

$$\lim_{t \rightarrow 0} t^{\gamma/2} \|e^{t\Delta} f\|_{\dot{B}_{r,q_1}^{\alpha+\beta}} = 0$$

provided $\sigma > 0$.

3. PROOF OF THEOREM 1

In this section we establish an *a priori* Besov space bound on solutions to the LANS equation. This argument requires qualitatively different methods than those used in [4] and [7] to obtain comparable bounds in the Sobolev space norm. Moreover, the result applies in dimension $n \geq 3$, an improvement over the requirement that $n = 3$ for Sobolev spaces.

We begin by recalling Bony's notion of paraproduct (see 0.17 in Chapter 2 of [11]). We have that $fg = T_f g + T_g f + R(f, g)$, where

$$(3.1) \quad T_f g = \sum_k (S_{k-2} f) \Delta_k g$$

and

$$R(f, g) = \sum_k \left(\sum_{l=k-1}^{k+1} \Delta_l f \right) (\Delta_k g).$$

The following calculation will be of use throughout this section. Using Bony's paraproduct and equations (2.1) and (2.2), we have that

$$\begin{aligned}\Delta_j(fg) &= \sum_{|j-k|\leq 2} \Delta_j(S_{k-2}f\Delta_k g) + \sum_{|j-k|\leq 2} \Delta_j(S_{k-2}g\Delta_k f) \\ &\quad + \sum_{k\geq j-3} \Delta_j(\Delta_k g \sum_{l=k-1}^{k+1} \Delta_l f) \\ &= I + II + III.\end{aligned}$$

Applying Young's inequality and then Holder's inequality, we have

$$\begin{aligned}(3.2) \quad \|I\|_p &\leq C \sum_{|j-k|\leq 2} \|\psi_j\|_1 \|S_{k-2}f\Delta_k g\|_p \\ &\leq C \sum_{|j-k|\leq 2} \|S_{k-2}f\|_\infty \|\Delta_k g\|_p,\end{aligned}$$

and similarly

$$(3.3) \quad \|II\|_p \leq C \sum_{|j-k|\leq 2} \|S_{k-2}g\|_\infty \|\Delta_k f\|_p.$$

$$(3.4) \quad \|III\|_p \leq C \sum_{k\geq j-3} \left(\sum_{l=k-1}^{k+1} \|\Delta_k g\|_p \|\Delta_l f\|_\infty \right).$$

We pause here to comment on our strategy. To prove Theorem 1, we need bounds on the two separate pieces of the Besov space norm. We will begin by working with the more complicated $\|\cdot\|_{\tilde{B}_{p,q}^s}$ piece.

The following proposition is the central piece to our desired estimate.

Proposition 4. *Any solution u to (1.1) satisfies*

$$\frac{d}{dt} \|u\|_{\tilde{B}_{2,q}^r}^q \leq C \|u\|_{\tilde{B}_{2,q}^{1+n/2}} \|u\|_{\tilde{B}_{2,q}^r}^q,$$

provided $r > 2$.

The proof of the Proposition is the content of the next Section.

3.1. Proof of Proposition 4. We begin with an equivalent form of the LANS equations (see Section 3 of [4]):

$$\begin{aligned}&\partial_t(1 - \alpha^2 \Delta)u + \nabla_u(1 - \alpha^2 \Delta)u - \alpha^2(\nabla u)^T \cdot \Delta u \\ &= -\nu(1 - \alpha^2 \Delta)P^\alpha \Delta u - \nabla p,\end{aligned}$$

Before proceeding, we recall the definitions of some of these objects. First, $\nabla_u v = (v \cdot \nabla)u$ is the vector with j^{th} component $\sum_{i=1}^n v_i \partial_{x_i} u_j$. Secondly, $(\nabla u)^T \cdot v$ is the vector with j^{th} component equal to the \mathbb{R}^n dot product of the vectors $\partial_j u$ and v .

Now, applying Δ_j to both sides and taking the L^2 inner product with $2\Delta_j u$, we get

$$(3.5) \quad I_1 + I_2 + I_3 + I_4 = I_5$$

where

$$\begin{aligned} I_1 &= (\partial_t(1 - \alpha^2 \Delta) \Delta_j u, \Delta_j u)_{L^2}, \\ I_2 &= (\Delta_j(\nabla_u(1 - \alpha^2 \Delta)u, \Delta_j u)_{L^2}, \\ I_3 &= -\alpha^2(\Delta_j((\nabla u)^T \cdot \Delta u, \Delta_j u)_{L^2}, \\ I_4 &= \nu((1 - \alpha^2 \Delta)P^\alpha \Delta \Delta_j u, \Delta_j u)_{L^2}, \\ I_5 &= -(\nabla \Delta_j p, \Delta_j u)_{L^2}. \end{aligned}$$

Applying integration by parts to I_1 , I_4 , and I_5 , we get

$$(3.6) \quad \frac{1}{2} \frac{d}{dt} (\|\Delta_j u\|_{L^2}^2 + \alpha^2 \|A^{1/2} \Delta_j u\|_{L^2}^2) \leq |I_2| + |I_3|.$$

To proceed, we need to estimate I_2 and I_3 .

Step 1: I_2 and I_3 estimates. We begin with I_2 , and we have by Holder's inequality that

$$|I_2| \leq C \|\Delta_j u\|_2 \|\Delta_j(\nabla_u(1 - \alpha^2 \Delta)u)\|_2.$$

To estimate the second term, we use (3.2), (3.3), and (3.4) to write

$$\Delta_j(\nabla_u(1 - \alpha^2 \Delta)u) = J_1 + J_2 + J_3$$

where

$$\begin{aligned} J_1 &= \sum_{|j-k| \leq 2} \Delta_j((S_{k-2}u \cdot \nabla) \Delta_k u) - \alpha^2 \sum_{|j-k| \leq 2} \Delta_j((S_{k-2}u \cdot \nabla) \Delta_k \Delta u) \\ J_2 &= \sum_{|j-k| \leq 2} \Delta_j((\Delta_k u \cdot \nabla) S_{k-2}u) - \alpha^2 \sum_{|j-k| \leq 2} \Delta_j((\Delta_k u \cdot \nabla) S_{k-2} \Delta u) \\ J_3 &= \sum_{k \geq j-3} \Delta_j \left(\Delta_k u \sum_{l=-1}^{l=1} \nabla \Delta_{k+l} u \right) - \alpha^2 \sum_{k \geq j-3} \Delta_j \left(\Delta_k u \sum_{l=-1}^{l=1} \nabla \Delta_{k+l} \Delta u \right). \end{aligned}$$

For notational convenience, we define $J_{i,j}$, with $j = 1, 2$, to be the j^{th} of the two summations in J_i . We begin with $J_{1,2}$. Recalling that ψ_j is the convolution kernel for Δ_j , we have by integration by parts and the incompressibility condition that

$$\begin{aligned} &\int \psi_j(x-y)((S_{k-2}u(y)) \nabla (\Delta \Delta_k u(y))) dy \\ &= \int -\Delta(\nabla \psi_j(x-y) \cdot (S_{k-2}u(y))) \Delta_k u(y) dy. \end{aligned}$$

Next, we use the product rule to distribute the Laplacian through the product, apply Young's inequality, and then take the L^∞ norm of the pieces involving $S_{k-2}u$ and its

derivatives. Recalling that the L^1 norm of ψ_j and its derivatives are independent of j , we have

$$\begin{aligned} & \left\| \sum_{|j-k|\leq 2} \Delta_j((S_{k-2}u \cdot \nabla) \Delta_k \Delta u) \right\|_{L^2} \\ & \leq C \sum_{|j-k|\leq 2} (\|S_{k-2}u\|_{L^\infty} + \|\nabla S_{k-2}u\|_{L^\infty} + \|\Delta S_{k-2}u\|_{L^\infty}) \|\Delta_k u\|_{L^2}. \end{aligned}$$

Recalling the definition of S_{k-2} and using Bernstein's inequality (equation (2.4)) we get

$$\begin{aligned} & \left\| \sum_{|j-k|\leq 2} \Delta_j((S_{k-2}u \cdot \nabla) \Delta_k \Delta u) \right\|_{L^2} \\ & \leq C \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{(2+n/2)m} \|\Delta_m u\|_{L^2}. \end{aligned}$$

For $J_{1,1}$, a similar computation gives

$$\begin{aligned} & \left\| \Delta_j((S_{k-2}u \cdot \nabla) \Delta_k u) \right\|_{L^2} \\ & \leq C \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{nm/2} \|\Delta_m u\|_{L^2}. \end{aligned}$$

So we finally get that J_1 satisfies

$$|J_1| \leq C \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{(2+n/2)m} \|\Delta_m u\|_{L^2}.$$

J_2 satisfies the same estimate, so we turn to J_3 . Again using integration by parts, Young's inequality, and then Bernstein's inequality, we have

$$\|J_{3,2}\|_{L^2} \leq C \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(2+n/p)(k+l)} \|\Delta_{k+l} u\|_{L^2}$$

and

$$\|J_{3,1}\|_{L^2} \leq C \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(n/p)(k+l)j} \|\Delta_{k+l} u\|_{L^2}.$$

So J_3 satisfies

$$|J_3| \leq C \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(2+n/p)(k+l)} \|\Delta_{k+l} u\|_{L^2}.$$

So we finally estimate I_2 by

$$\begin{aligned} (3.7) \quad |I_2| &= C \|\Delta_j u\|_{L^2} \|J_1 + J_2 + J_3\|_{L^2} \\ &\leq C \|\Delta_j u\|_{L^2} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{(2+n/2)m} \|\Delta_m u\|_{L^2} \\ &\quad + C \|\Delta_j u\|_{L^2} \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(2+n/p)(k+l)} \|\Delta_{k+l} u\|_{L^2}. \end{aligned}$$

The estimation of I_3 is similar, so the details will be omitted. The key difference between the two estimates is that, in the case of I_2 , integrating the gradient term by parts and applying the incompressibility condition essentially removed one of the three derivatives from I_2 . Because the gradient term in I_3 is $(\nabla u)^T$ instead of ∇u , this does not work when estimating I_3 , and the result is the presence of an extra derivative. The estimate for I_3 is

$$(3.8) \quad \begin{aligned} |I_3| &\leq C \|\Delta_j u\|_{L^2} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{(3+n/2)m} \|\Delta_m u\|_{L^2} \\ &\quad + C \|\Delta_j u\|_{L^2} \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(3+n/p)(k+l)} \|\Delta_{k+l} u\|_{L^2}. \end{aligned}$$

Using (3.7) and (3.8) in (3.6) we get

$$(3.9) \quad \begin{aligned} &\frac{d}{dt} (\|\Delta_j u\|_{L^2}^2 + \alpha^2 \|A^{1/2} \Delta_j u\|_{L^2}^2) \\ &\leq C \|\Delta_j u\|_{L^2} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{(3+n/2)m} \|\Delta_m u\|_{L^2} \\ &\quad + C \|\Delta_j u\|_{L^2} \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(3+n/p)(k+l)} \|\Delta_{k+l} u\|_{L^2}. \end{aligned}$$

In the next section, we work on the left-hand side of (3.9).

Step 2: Exploiting the LANS term. By Plancherel's Theorem (3.9) becomes

$$\begin{aligned} 2^{2j} \frac{d}{dt} (\|\Delta_j u\|_{L^2}^2) &\leq C \|\Delta_j u\|_{L^2} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{(3+n/2)m} \|\Delta_m u\|_{L^2} \\ &\quad + C \|\Delta_j u\|_{L^2} \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(3+n/2)(k+l)} \|\Delta_{k+l} u\|_{L^2}. \end{aligned}$$

Computing the time derivative, multiplying both sides by $q 2^{rjq} \|\Delta_j u\|_{L^2}^{q-1}$ and summing over $j \geq 0$, we get

$$(3.10) \quad \frac{d}{dt} \|u\|_{\tilde{B}_{2,q}^r}^q \leq K_1 + K_2,$$

where

$$K_1 = Cq \sum_{j \geq 0} 2^{rjq} 2^{-2j} \|\Delta_j u\|_{L^2}^{q-1} \sum_{|j-k|\leq 2} \|\Delta_k u\|_{L^2} \sum_{m < k-2} 2^{(3+n/2)m} \|\Delta_m u\|_{L^2},$$

and

$$K_2 = Cq \sum_{j \geq 0} 2^{rjq} 2^{-2j} \|\Delta_j u\|_{L^2}^{q-1} \sum_{k \geq j-3} \|\Delta_k u\|_{L^2} \sum_{l=-1}^{l=1} 2^{(3+n/p)(k+l)} \|\Delta_{k+l} u\|_{L^2}.$$

We remark that the 2^{-2j} term in K_1 and K_2 is the “gain” we get by using the LANS equations instead of the Navier-Stokes equations.

Step 3: Estimating K_1 and K_2 . We begin by re-writing K_1 as

$$K_1 = Cq \sum_{j \geq 0} 2^{rjq} \|\Delta_j u\|_{L^2}^{q-1} \sum_{k=-2}^2 \|\Delta_{j+k} u\|_{L^2} \sum_{m < j+k-2} 2^{(3+n/2)m-2j} \|\Delta_m u\|_{L^2}.$$

We start by working on the last summation. We have by Holder's inequality that

$$\begin{aligned} & \sum_{m < j+k-2} 2^{(1+n/2)m} 2^{2m-2j} \|\Delta_m u\|_2 \\ & \leq C \left(\sum_{m < j+k-2} 2^{(1+n/2)qm} \|\Delta_m u\|_2^q \right)^{1/q} \left(\sum_{m < j+k-2} 2^{2q'(m-(j+k-2))} 2^{2(k-2)q'} \right)^{1/q'} \\ & \leq C \|u\|_{\tilde{B}_{2,q}^{1+n/2}}, \end{aligned}$$

where q' is the Holder conjugate exponent to q .

Returning to K_1 , we have

$$|K_1| \leq C \|u\|_{\tilde{B}_{2,q}^{1+n/2}} \sum_{j \geq 0} 2^{rjq} \|\Delta_j u\|_{L^2}^q \sum_{k=-2}^2 \|\Delta_{j+k} u\|_{L^2}.$$

For the remaining summation, we have

$$\begin{aligned} & \sum_{j \geq 0} 2^{rjq} \|\Delta_j u\|_{L^2}^{q-1} \sum_{k=-2}^2 \|\Delta_{j+k} u\|_{L^2} \\ & \leq C \left(\sum_{j \geq 0} 2^{rj(q-1)q'} \|\Delta_j u\|_{L^2}^{q'(q-1)} \right)^{q'} \left(\sum_j \left(\sum_{k=-2}^2 2^{r(j+k)} 2^{-rk} \|\Delta_{j+k} u\|_{L^2} \right)^q \right)^q \\ & \leq C \|u\|_{\tilde{B}_{2,q}^r}^{q-1} \|u\|_{\tilde{B}_{2,q}^r} \leq C \|u\|_{\tilde{B}_{2,q}^r}^q, \end{aligned}$$

where q' is again the Holder conjugate exponent to q . So we finally bound K_1 by

$$K_1 \leq C \|u\|_{\tilde{B}_{2,q}^r}^q \|u\|_{\tilde{B}_{2,q}^{1+n/2}}.$$

Bounding K_2 the same way, we have

$$(3.11) \quad |K_2| \leq C \|u\|_{\tilde{B}_{2,q}^r}^{q-1} \tilde{K},$$

where

$$\tilde{K} = \left(\sum_j \left(\sum_{k > -3} \sum_{l=-1}^1 2^{(r-2)j} 2^{(3+n/2)(j+k+l)} \|\Delta_{j+k} u\|_{L^2} \|\Delta_{j+k+l} u\|_{L^2} \right)^q \right)^{1/q}.$$

A straightforward calculation shows that \tilde{K} satisfies

$$\tilde{K} \leq C \|u\|_{\tilde{B}_{2,q}^{1+n/2}} \left(\sum_j \left(\sum_{k > -3} 2^{k(2-r)} \sum_{l=-1}^1 (2^{r(j+k)} \|\Delta_{j+k} u\|_2) \right)^q \right)^{1/q}.$$

Applying Minkowski's inequality, we get

$$(3.12) \quad |\tilde{K}| \leq C \|u\|_{\tilde{B}_{2,q}^{1+n/2}} \|u\|_{\tilde{B}_{2,q}^r} \sum_{k>-3} 2^{k(2-r)}.$$

This last sum will be finite provided $r > 2$. We remark that this restriction on r could be lifted by allowing for additional regularity in the $B_{2,q}^{1+n/2}$ term. Specifically, by choosing ε such that $2 < r + \varepsilon$, we would have

$$|\tilde{K}| \leq \|u\|_{\tilde{B}_{2,q}^{1+n/2+\varepsilon}} \|u\|_{\tilde{B}_{2,q}^r}.$$

Using this estimate would provide a slightly different result for Proposition 4.

Assuming $r > 2$, we plug these estimates back into the K_2 estimate, and get

$$|K_2| \leq C \|u\|_{\tilde{B}_{2,q}^r}^q \|u\|_{\tilde{B}_{2,q}^{1+n/p}}$$

provided $r > 2$.

Plugging the K_1 and K_2 estimates into (3.10), we finally get

$$(3.13) \quad \frac{d}{dt} \|u\|_{\tilde{B}_{2,q}^r}^q \leq C \|u\|_{\tilde{B}_{2,q}^r}^q \|u\|_{\tilde{B}_{2,q}^{1+n/2}}$$

for $r > 2$.

3.2. Proof of Theorem 1. Now we return to proving Theorem 1. Recalling our construction of Besov spaces, we need to show that

$$(3.14) \quad \|\Psi * u(t)\|_{L^2} \leq \|u_0\|_{B_{2,q}^r} \exp \left(C \int_I \|u(t)\|_{B_{2,q}^{1+n/2}} dt \right)$$

and

$$(3.15) \quad \|u(t)\|_{\tilde{B}_{2,q}^{n/2}} \leq C \|u_0\|_{B_{2,q}^r} \exp \left(C \int_I \|u(t)\|_{B_{2,q}^{1+n/2}} dt \right).$$

To prove (3.15), we re-write (3.13) as

$$(3.16) \quad \frac{d}{dt} \|u\|_{\tilde{B}_{2,q}^r} \leq C \|u\|_{\tilde{B}_{2,q}^r} \|u\|_{\tilde{B}_{2,q}^{1+n/2}}$$

Applying Gronwall's inequality to (3.16), we get

$$(3.17) \quad \begin{aligned} \|u(t)\|_{\tilde{B}_{2,q}^r} &\leq \|u_0\|_{\tilde{B}_{2,q}^r} \exp \left(C \int_0^T \|u(s)\|_{\tilde{B}_{2,q}^{1+n/2}} ds \right) \\ &\leq \|u_0\|_{B_{2,q}^r} \exp \left(C \int_0^T \|u(s)\|_{B_{2,q}^{1+n/2}} ds \right) \end{aligned}$$

which proves (3.15).

To prove (3.14), we go back to the LANS equation (see Section 3 of [4]):

$$(3.18) \quad \begin{aligned} &\partial_t (1 - \alpha^2 \Delta) u + \nabla_u [(1 - \alpha^2 \Delta) u] - \alpha^2 (\nabla u)^T \cdot \Delta u \\ &= - (1 - \alpha^2 \Delta) A u - \nabla p. \end{aligned}$$

Taking the L^2 product of (3.18) with u , we get

$$I_1 + I_2 + I_3 = J_1 + J_2$$

where

$$\begin{aligned} I_1 &= (\partial_t(1 - \alpha^2 \Delta)u, u) \\ I_2 &= (\nabla_u u, u) \\ I_3 &= -\alpha^2 ((\nabla_u \Delta u, u) + ((\nabla u)^T \cdot \Delta u, u)) \\ J_1 &= -((1 - \alpha^2 \Delta)(Au), u) \\ J_2 &= (\nabla p, u). \end{aligned}$$

We start with I_1 , which becomes

$$\begin{aligned} I_1 &= (\partial_t u, u) - \alpha^2 (\Delta \partial_t u, u) \\ &= \frac{1}{2} \partial_t (\|u\|_{L^2}^2 + \alpha^2 \|A^{1/2} u\|_{L^2}^2). \end{aligned}$$

Applying integration by parts to I_2 , I_3 , and J_2 and recalling that $\operatorname{div} u = 0$, we get that all three terms vanish. For J_1 we have

$$J_1 = -((1 - \alpha^2 \Delta)(Au), u) = -(A^{1/2} u, A^{1/2} u) - \alpha^2 (Au, Au).$$

Applying this to (3.18), we get

$$(3.19) \quad \frac{1}{2} \partial_t (\|u(t)\|_{L^2}^2 + \alpha^2 \|u(t)\|_{\dot{H}^{1,2}}^2) \leq -(\|A^{1/2} u(t)\|_{L^2}^2 + \alpha^2 \|Au(t)\|_{L^2}^2),$$

where \dot{H} denotes the homogeneous Sobolev norm. This proves that $\|u(t)\|_{H^{1,2}}$ is decreasing in time.

Using (3.19), we have

$$(3.20) \quad \|\Psi * u(t)\|_{L^2} \leq \|u(t)\|_{H^{1,2}} \leq M,$$

which is stronger than (3.14). This proves Theorem 1.

4. LOCAL SOLUTIONS IN $\dot{C}_{a;s,p,q}^T$

We begin by re-writing the LANS equation as

$$(4.1) \quad \partial_t u - Au + P^\alpha (\operatorname{div} \cdot (u \otimes u) + \operatorname{div} \tau^\alpha u) = 0,$$

where the recurring terms are as in (1.1), with the exception that we set $\nu = 1$. For the new terms, we set $A = P^\alpha \Delta$, $u \otimes u$ is the tensor with jk -components $u_j u_k$ and $\operatorname{div} \cdot (u \otimes u)$ is the vector with j -component $\sum_k \partial_k (u_j u_k)$. P^α is the Stokes Projector, defined as

$$P^\alpha(w) = w - (1 - \alpha^2 \Delta)^{-1} \nabla f$$

where f is a solution of the Stokes problem: Given w , there is a unique divergence-free v and a unique (up to additive constants) function f such that

$$(1 - \alpha^2 \Delta)v + \nabla f = (1 - \alpha^2 \Delta)w.$$

For a more explicit treatment of the Stokes Projector, see Theorem 4 of [9].

Using Duhamel's principle, we write (4.1) as the integral equation

$$(4.2) \quad u = \Gamma \varphi - G \cdot P^\alpha (\operatorname{div} (u \otimes u + \tau^\alpha(u)))$$

with

$$(\Gamma \varphi)(t) = e^{tA} \varphi,$$

where A agrees with Δ when restricted to $P^\alpha H^{r,p}$, and

$$G \cdot g(t) = \int_0^t e^{(t-s)A} \cdot g(s) ds.$$

To prove Theorem 2 and Theorem 4, we will apply the standard contraction method. Section 4.1 contains the supporting details necessary for the proof, which can be found in Section 4.2. These results follow the same outline as that from [7], and more detailed arguments can be found there.

4.1. Preliminary Estimates. Our first calculation is a result for τ^α .

Lemma 1. *Given $r > 1$, $1 \leq q < \infty$ and $p, \bar{p} \in (1, \infty)$, where*

$$\begin{aligned} p &\leq 2\bar{p}, \\ \bar{s} &:= n(2/p - 1/\bar{p}), \\ 0 &\leq \bar{s} < r - 1, \end{aligned}$$

we have $\operatorname{div} \tau^\alpha : B_{p,q}^r \rightarrow B_{\bar{p},q}^r$. Specifically, we have the estimate

$$\|\operatorname{div} \tau^\alpha(u)\|_{B_{\bar{p},q}^r} \leq C \|u\|_{B_{p,q}^r}^2.$$

We have by Proposition 2 that

$$\begin{aligned} \|\operatorname{div} \tau^\alpha(u)\|_{B_{\bar{p},q}^r} &\leq C \|\tau^\alpha(u)\|_{B_{\bar{p},q}^{r+1}} \\ &\leq C \|Def(u) \cdot Rot(u)\|_{B_{\bar{p},q}^{r-1}} \\ &\leq C \|\nabla u\|_{B_{p,q}^{r-1}}^2 \leq C \|u\|_{B_{p,q}^r}^2. \end{aligned}$$

We remark that this differs from the Sobolev result in that $r > 1$ instead of $r \geq 1$ and that $\bar{s} < r - 1$ instead of $\bar{s} \leq r - 1$. This Lemma has an immediate corollary.

Corollary 3. *$\operatorname{div} \tau^\alpha : \dot{C}_{a;r,p,q}^T \rightarrow \dot{C}_{2a;r,\bar{p},q}^T$, with the estimate*

$$\|\operatorname{div} \tau^\alpha(u)\|_{2a;r,\bar{p},q} \leq C \|u\|_{a;r,p,q}^2.$$

Next, we record some results for the operator V^α .

Proposition 5. *With the parameters as in Lemma 1, we have*

$$V^\alpha : \dot{C}_{a;r,p,q}^T \times \dot{C}_{a;r,p,q}^T \rightarrow \dot{C}_{2a;r-1,p',q}^T$$

with the estimate

$$(4.3) \quad \|V^\alpha(u, v)\|_{2a;s-1,p',q} \leq \|u\|_{a;s,p,q} \|v\|_{a;s,p,q}.$$

This follows from a calculation parallel to the one used to prove Proposition 1 in [7] with (2.3) replacing the Sobolev embedding results.

Corollary 4. *With the same assumptions on the parameters as in Proposition 5, we have that*

$$(4.4) \quad \begin{aligned} &\|V^\alpha(u(s)) - V^\alpha(v(s))\|_{a;s-1,p',q} \\ &\leq C(\|u\|_{a/2;s,p,q} + \|v\|_{a/2;s,p,q}) \|u - v\|_{a/2;s,p,q}. \end{aligned}$$

For the remainder of the section, we impose the following list of restrictions on our parameters. We have

$$(4.5) \quad \begin{aligned} -\infty &< s_0 \leq s_1 < \infty \\ 1 &\leq q \leq \infty \\ 1 &\leq p_0 \leq p_1 < \infty \\ \sigma &= s_1 - s_0 + n(1/p_0 - 1/p_1). \end{aligned}$$

The following proposition is an immediate consequence of Corollary 2.

Proposition 6. *Provided the parameters satisfy (4.5), we have that $\Gamma : B_{p_0,q}^{s_0} \rightarrow \dot{C}_{\sigma/2;s_1,p_1,q}^T$.*

Lastly, we turn our attention to the operator G . Using Proposition 3, we have

$$\begin{aligned} \|G \cdot u\|_{B_{p_1,q}^{s_1}} &\leq C \int_0^t (t-s)^{-\sigma/2} \|u\|_{B_{p_0,q}^{s_0}} ds \\ &\leq C \|u\|_{k;s_0,p_0,q} \int_0^t (t-s)^{-\sigma/2} t^{-k} \\ &\leq C t^{-\sigma/2-k+1} \|u\|_{k;s_0,p_0,q}. \end{aligned}$$

We record this as a proposition.

Proposition 7. *With parameters as specified in (4.5), $0 < \sigma/2 < 1$, $0 \leq k_0 < 1$, and $k_1 = k_0 + \sigma/2 - 1$, we have*

$$\|G \cdot u\|_{k_1;s_1,p_1,q} \leq C \|u\|_{k_0;s_0,p_0,q}.$$

4.2. Proof of Theorems 2 and 4. As usual, we begin with the nonlinear map

$$\Phi u = \Gamma \varphi - G \cdot P^\alpha(V^\alpha(u)),$$

initial data $u_0 \in B_{p,q}^r$ and define

$$E_{T,M} = \{v \in \bar{C}_{r,p,q} \cap \dot{C}_{a;s,\tilde{p},q}^T : \|v - \Gamma \varphi\|_{0;r,p,q} + \|v\|_{a;s,\tilde{p},q} \leq M\}.$$

The proof is a standard contraction mapping argument. For the details, see [7]. We get that Φ will be a contraction on $E_{T,M}$ provided there exists a $\bar{b} > 0$ such that the following list of conditions holds (where, for notational convenience, we set $\bar{s} = s - 2 + \bar{b} - r + n/p$):

$$\begin{aligned}
1 &< p \leq \tilde{p} \\
1 &\leq q \leq \infty \\
s &> 1, \bar{b} \geq 1, \\
\bar{s}\tilde{p} &< n \\
0 &< 2a = s - n/\tilde{p} - r + n/p < 1 \\
0 &\leq \bar{s} < s - 1 \\
1 &< \frac{n\tilde{p}}{2n - \bar{s}\tilde{p}} < \infty \\
1 &> \bar{b} - r + n/p \\
1 &\leq \bar{b} + \frac{n}{\tilde{p}} - \bar{s} < 2 \\
2 - 2\bar{b} + \bar{s} &\leq \frac{n}{p} \leq 2 - \bar{b} + \bar{s}.
\end{aligned}$$

We remark that the requirements $s > 1$, $\bar{s} < s - 1$, and $1 > \bar{b} - r + n/p$ differ from the analogous Sobolev space requirements, which were $s \geq 1$, $\bar{s} \leq s - 1$, and $1 \geq \bar{b} - r + n/p$ respectively. These relations combine to force $r > n/p$.

This result is actually stronger than required for Theorem 4, but requires the presence of the artificial parameter \bar{b} (though this is somewhat compensated for by the explicit formula for a). To get the result of Theorem 4, we fix $\bar{b} = 1$, which provides the minimal allowable r (and we again set $\bar{s} = s - 2 + \bar{b} - r + n/p$):

$$\begin{aligned}
1 &< p \leq \tilde{p} \\
1 &\leq q \leq \infty \\
s &> 1, \bar{s}\tilde{p} < n \\
r &> n/p \\
0 &< 2a = s - n/\tilde{p} - b < 1 \\
0 &\leq \bar{s} < s - 1 \\
1 &< \frac{n\tilde{p}}{2n - \bar{s}\tilde{p}c} < \infty \\
0 &\leq \frac{n}{\tilde{p}} - \bar{s} < 1 \\
\bar{s} &\leq \frac{n}{p} \leq 1 + \bar{s}.
\end{aligned}
\tag{4.6}$$

As in Section 3 of [7], this general method can be modified to obtain the results in Theorem 2. We also have the following result which extends the range of parameters allowed on the auxiliary space.

Proposition 8. *Let $u_0 \in B_{p,q}^{s_1}(\mathbb{R}^n)$ be divergence-free, and let $s_1 < 1$. Let u be a solution to the LANS equation (1.1) such that*

$$u \in BC([0, T) : B_{p,q}^{s_1}(\mathbb{R}^n)) \cap \dot{C}_{(s_2-s_1)/2; s_2, p, q}^T,$$

where $0 < s_2 - s_1 < 1$ and $s_2 \geq 1$. Then for all $r \geq s_2$, we have that $u \in \dot{C}_{(r-s_1)/2; r, p, q}^T$.

We remark that the requirement $s_1 < 1$ can be removed and the assumption that $u \in BC([0, T] : B_{p, q}^{s_1}(\mathbb{R}^n)) \cap \dot{C}_{(s_2-s_1)/2; s_2, p, q}^T$ can be changed to $u \in BC([0, T] : B_{\tilde{p}, q}^{s_1}(\mathbb{R}^n)) \cap \dot{C}_{a; s_2, \tilde{p}, q}^T$ for $\tilde{p} > p$.

5. LOCAL SOLUTIONS IN $L^a((0, T) : B_{p, q}^s(\mathbb{R}^n))$

In this section we consider Theorems 3 and 5. Section 5.1 contains the necessary supporting estimates, and the theorems are proven in Section 5.2.

5.1. Preliminary Results. In this section we establish integral-in-time results for Besov space. The proofs are similar to those in [7] used for the analogous operators.

Proposition 9. *Let $1 < p_0 \leq p_1 < \infty$, $1 \leq q < \infty$, $-\infty < s_0 \leq s_1 < \infty$, and assume $0 < (s_1 - s_0 + n/p_0 - n/p_1)/2 = 1/\sigma$. Then Γ maps $B_{p_0, q_0}^{s_0}$ continuously into $L^\sigma((0, \infty) : B_{p_1, q_1}^{s_1})$ with the estimate*

$$\|\Gamma u\|_{L^\sigma((0, \infty) : B_{p_1, q_1}^{s_1})} \leq C \|u\|_{B_{p_0, q_0}^{s_0}}.$$

The proof is similar to Proposition 4 in [7], with two main distinctions, both due to the differences in interpolation theory between Sobolev and Besov spaces. The first is that we interpolate using s_0 instead of p_0 . The second difference is that we do not require $p_0 \leq \sigma$, as we did in Proposition 4 of [7].

Our next two results involve the operator V^α .

Proposition 10. *With the parameters s, p, p' and q as in Proposition 5, we have*

$$V^\alpha : L^\sigma((0, T) : B_{p, q}^s) \rightarrow L^{\sigma/2}((0, T) : B_{p, q}^{s-1})$$

with the estimate

$$\left(\int_0^T \|V^\alpha(u(s))\|_{B_{p', q}^{s-1}}^{\sigma/2} ds \right)^{2/\sigma} \leq \left(\int_0^T \|u(s)\|_{B_{p, q}^s}^\sigma ds \right)^{2/\sigma}.$$

Corollary 5. *If $u, v \in L^\sigma((0, T) : B_{p, q}^s)$, then*

$$\begin{aligned} & \left(\int_0^T \|V^\alpha(u(s)) - V^\alpha(v(s))\|_{B_{p, q}^{s-1}}^{\sigma/2} ds \right)^{2/\sigma} \\ & \leq \left(\int_0^T \|v(s) + u(s)\|_{B_{p, q}^s}^\sigma ds \right)^{2/\sigma} \left(\int_0^T \|v(s) - u(s)\|_{B_{p, q}^s}^\sigma ds \right)^{2/\sigma}. \end{aligned}$$

We conclude this section with estimates for G .

Proposition 11. *Given $1 \leq p_0 \leq p_1 < \infty$, $1 \leq q < \infty$, $-\infty < s_0 \leq s_1 < \infty$, $1 < \sigma_0 < \sigma_1 < \infty$ and $1/\sigma_0 - 1/\sigma_1 = 1 - (s_1 - s_0 + n/p_0 - n/p_1)/2$, for any $T \in (0, \infty]$, G sends $L^{\sigma_0}((0, T) : B_{p_0, q_0}^{s_0})$ into $L^{\sigma_1}((0, T) : B_{p_1, q_1}^{s_1})$ with the estimate*

$$\|G \cdot u\|_{L^{\sigma_1}((0, T) : B_{p_1, q_1}^{s_1})} \leq C \|u\|_{L^{\sigma_0}((0, T) : B_{p_0, q_0}^{s_0})}.$$

Proposition 12. $1 < p_0 \leq p_1 < \infty$, $1 \leq q < \infty$, $-\infty < s_0 \leq s_1 < \infty$, and assume $1/p_1 \leq 1/\sigma = 1 - (s_1 - s_0 + n/p_0 - n/p_1)/2 =$. Then G maps $L^\sigma((0, T) : B_{p_0, q_0}^{s_0})$ continuously into $BC([0, T] : B_{p_1, q_1}^{s_1})$ with the estimate

$$\sup_{t \in [0, T]} \|G \cdot u(t)\|_{B_{p_1, q_1}^{s_1}} \leq C \|u\|_{L^\sigma((0, T) : B_{p_0, q_0}^{s_0})}.$$

5.2. Proof of Theorems 3 and 5. As usual, we begin by defining a Banach space $X_{T, M}$ to be the set of all $u \in BC([0, T] : B_{p, q}^r) \cap L^\sigma((0, T) : B_{p, q}^s)$ such that

$$\sup_t \|u(t) - \Gamma u_0\|_{B_{p, q}^r} + \|u\|_{\sigma; s, \tilde{p}, q} \leq M$$

and we define the operator Φ by

$$\Phi u(t) = \Gamma \varphi + G(V^\alpha(u(t)))$$

where $\varphi \in B_{p, q}^r$.

The required restrictions on the parameters are similar to those from Section 4 of [7], and again we get that Φ will be a contraction provided there exists a $\bar{b} > 0$ such that the following holds (for notational convenience, we set $\bar{s} = s - 2 + \bar{b} - r + n/p$): We record these parameters:

$$\begin{aligned} 1 &< p \leq \tilde{p} < \infty \\ 1 &\leq q \leq \infty \\ s &> 1, \quad \bar{b} \geq 1, \quad \bar{s}\tilde{p} < n \\ 0 &< 2/\sigma = s - n/\tilde{p} - b < 1 \\ 0 &\leq \bar{s} < s - 1 \\ 1 &< \frac{n\tilde{p}}{2n - \bar{s}\tilde{p}} < \infty \\ 1 &> \bar{b} - r + n/p \\ s - \bar{b} &\leq \frac{n}{\tilde{p}} + r - n/p \leq s \\ \sigma/2 &\leq p \leq \sigma. \end{aligned}$$

As in Section 4, we get Theorem 5 by choosing $\bar{b} = 1$:

$$\begin{aligned}
 (5.1) \quad & 1 < p \leq \tilde{p} < \infty \\
 & 1 \leq q \leq \infty \\
 & s > 1, \bar{s}\tilde{p} < n \\
 & 0 < 2/\sigma = s - n/\tilde{p} - b < 1 \\
 & 0 \leq \bar{s} < s - 1 \\
 & 1 < \frac{n\tilde{p}}{2n - \bar{s}\tilde{p}} < \infty \\
 & r > n/p \\
 & s - 1 \leq \frac{n}{\tilde{p}} + r - n/p \leq s \\
 & \sigma/2 \leq p \leq \sigma.
 \end{aligned}$$

As in Section 4 of [7], we can also obtain Theorem 3 by modifying the standard method to fit the specific cases. We also have the following result which extends the range of parameters allowed on the auxiliary space.

Proposition 13. *Let $k > s_2 > s_1$, with $s_2 \geq 1$, and let ε be a small positive number. Then, for $k - s_2 = s_2 - s_1 = \varepsilon$, for any solution u to the LANS equation (1.1) where*

$$u \in BC([0, T) : B_{p,q}^{s_1}(\mathbb{R}^n) \cap L^{2/(s_2-s_1)}((0, T) : B_{p,q}^{s_2}(\mathbb{R}^n)),$$

we have that $u \in L^1((0, T) : B_{p,q}^k(\mathbb{R}^n))$ provided $s_2 \geq n/p$.

We remark that this result can be extended, with the additional requirement that, for k such that $k - s_1 \geq 2$, $u \in L^a((0, T) : B_{p,q}^k(\mathbb{R}^n))$ requires $\tilde{p} > p$.

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